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ON THE EXISTENCE OF ORDERED COUPLINGS OF RANDOM SETS — WITH APPLICATIONS

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ABSTRACT

Let ψ and φ be two given random closed sets in a locally compact second countable topological space S. (They need not be based on the same probability space.) The main result gives necessary and sufficient conditions on the distributions of ψ and φ, for the existence of two random closed sets $\hat{\psi}$ and $\hat{\varphi}$, based on the same probability space and such that their distributions coincide with those of ψ and φ, resp., and $\hat{\psi} \subseteq \hat{\varphi}$ a.s.

This coupling result tells us in particular when a probability distribution on S is selectionable w.r.t. (the distribution of) a random closed set. An existence result for realizable thinnings of a simple point process is obtained by specializing it to supports of random measures.

The coupling result is extended to random variables in a countably based continuous poset. As examples we mention various kinds of random capacities — in particular random measures — and random compact (saturated) sets. Moreover, the extended result tells us when a probability distribution on S is selectionable w.r.t. the distribution of a random compact (saturated) set.

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1. Introduction

We are concerned with the problem of finding an ordered coupling of two given random sets ψ and φ in a given topological space S. They need not be based on the same probability space. Denote by $\stackrel{d}{=}$ equality in distribution (i.e., law). By a **coupling** of ψ and φ we mean a pair $(\hat{\psi}, \hat{\varphi})$ of random subsets of S which are based on the same probability space, and such that $\hat{\psi} \stackrel{d}{=} \psi$ and $\hat{\varphi} \stackrel{d}{=} \varphi$. A non-interesting example of a coupling is obtained by taking $\hat{\psi}$, $\hat{\varphi}$ independent.

A coupling $(\hat{\psi}, \hat{\varphi})$ of ψ and φ will be said to be **ordered,** if $\hat{\psi} \subseteq \hat{\varphi}$ a.s. The existence of an ordered coupling of ψ and φ will be denoted $\psi \subseteq_{\mathrm{st}} \varphi$ — the subscript st indicating that it is a stochastic ordering, i.e., an ordering of probability distributions.

We now state our main result. In it, and subsequently, all underlying probability measures are denoted P. Note that we call a topological space **locally compact,** if every point has a neighborhood basis of compact sets. This definition may seem peculiar or unusual, but is easily recognized as the basic property of locally compact Hausdorff spaces.

THEOREM A: *et S be a locally compact, second countable topological space, and consider two random closed sets ψ and φ in S.*

(a) Then $\psi \subseteq_{\mathrm{st}} \varphi$ if, and only if, the inequality

$$P\bigcap_{i=1}^{n}\{\psi \cap B_i \neq \emptyset\} \leq P\bigcap_{i=1}^{n}\{\varphi \cap B_i \neq \emptyset\} \tag{1}$$

holds true for $n = 1, 2, \dots$ and open B_i.

(b) Suppose S in addition is sober. Then $\psi \subseteq_{\mathrm{st}} \varphi$ if, and only if, (1) *holds true for $n = 1, 2, \dots$ and compact B_i.*

(c) Suppose S in addition is Hausdorff. Then $\psi \subseteq_{\mathrm{st}} \varphi$ if, and only if, (1) *holds true for $n = 1, 2, \dots$ and closed B_i.*

Part (a) of the theorem depends only on the collection of open subsets of S. Thus that equivalence holds true whenever the topology of S is order isomorphic to the topology of a locally compact second countable space. This is known to hold for any set provided with a continuous countably generated topology (cf. Hofmann and Mislove [5] and the references therein). In Section 2. we have collected some background material on topology, continuous posets (i.e., partially ordered sets), random sets and random variables in continuous posets.

Let us only note at this point that Hausdorff spaces are sober, so that all three parts of the theorem hold true if S is a locally compact Polish space.

We give two proofs of Theorem A. The first is given in Section 3.. It is based on a particular case of a well-known theorem of Strassen [19], see Ligget [10, p. 72], which we state in Section 2.. Rather surprisingly it turns out that this particular case of Strassen's theorem is a straightforward consequence of Theorem A. So Theorem A and some of its corollaries are indeed equivalent to this particular case of Strassen's theorem. It is therefore of theoretical interest to prove Theorem A without appealing to Strassen's theorem. This is done in Section 5..

Theorem A has interesting applications, some of which we now outline. Refer to Section 4. for details. If S in addition to being locally compact and second countable, also is Hausdorff, then any simple point process ξ on S may be identified with its support

$$\operatorname{supp}\xi = \{s \in S : \xi\{s\} > 0\},$$

which is a locally finite random closed set in S. Moreover, if η is a simple point process, then $\eta \leq \xi$ (in the sense that $\eta(B) \leq \xi(B)$ for Borel sets $B \subseteq S$) if, and only if, $\operatorname{supp}\eta \subseteq \operatorname{supp}\xi$. So Theorem A also yields an existence theorem for so called realizable thinnings (cf. Rolski and Szekli [16, Definition 1]) of simple point processes. Our result, which we formulate in a larger generality for supports of random measures and mean is interesting from practical as well as theoretical considerations, seems to be new.

In general it is not enough in Theorem A to require (1) to hold for $n = 1$ only. Section 3. contains a simple counter example in which S is a two point set provided with the discrete topology.

However, if ψ equals the singleton closure of some random element ξ in S, a simple argument shows that (1) holds for all $n \geq 1$ if it holds for $n = 1$. By pursuing this, a characterization of the probability distributions on S that are selectionable w.r.t. (the distribution of) a given random closed set φ, is obtained. Our result, which generalizes a theorem of Artstein [1], requires S to be sober (in addition to being locally compact, second countable), because otherwise there need not be a random element $\hat{\xi}$ in S satisfying $\{\hat{\xi}\}^- = \hat{\psi}$, if $(\hat{\psi}, \hat{\varphi})$ is an ordered coupling of $\psi = \{\xi\}^-$ and φ, so satisfies $\hat{\psi} \subseteq \hat{\varphi}$ a.s.

Artstein (op. cit.) showed the characterization for locally compact Polish S.

Recently, Ross [17] extended it to Hausdorff regular spaces. The possibility to go beyond the Hausdorff setting in the locally compact case is interesting.

The existence result for selectionable distributions has a coupling theorem for random elements in S as a further corollary, since on every sober space there is a partial order (often called the specialization order), which in a certain sense is consistent with the topology. At first sight this coupling result may seem obvious and useless. However, it has the particular case of Strassen's theorem, which we used in our first proof of Theorem A, as an immediate consequence.

Another striking consequence of it is a coupling result for random variables in a countably generated continuous poset. This result extends Theorem A, since every random closed set φ in a space S provided with a countably generated continuous topology $\mathcal{G}$, may be identified with its complement $S \setminus \varphi$, which is a random variable in $\mathcal{G}$.

This coupling result for random variables in a continuous poset has other consequences than Theorem A. Among them we first note a known result on the stochastic ordering of random measures due to Rolski and Szekli [16, Theorem 1], which we prefer to formulate in the more general setting of random capacities. It is of no value to specialize this result further to simple point processes, since the already mentioned thinning theorem is much sharper.

Note next that a random compact set in a locally compact Polish space S is a random variable in its collection $\mathcal{K}$ of compact subsets — the latter being a countably generated continuous poset. So the coupling theorem for random variables in such posets, yields a characterization of $\psi \subseteq_{\rm st} \varphi$ when ψ and φ are random compact sets in S. This result may of course be applied to the case when ψ is of the form $\{\xi\}$. We then obtain Theorem 2.1 of Artstein [1], which tells us exactly when a probability distribution on S is selectionable w.r.t. φ.

Needless to say, we formulate these results also for non-Hausdorff spaces S.

Finally, in Section 5. we give an independent proof of a coupling theorem for random variables in a countably generated continuous lattice, from which Theorem A follows immediately (because as we already remarked, any random closed set may be identified with a random variable in a continuous lattice). This is our second proof of Theorem A.

2. Preliminaries, notations

Let S be a topological space. Denote by $\mathcal{G}$, $\mathcal{F}$ and $\mathcal{K}$ its collections of open, closed and compact subsets. Note that we do not assume S to be Hausdorff — thus cannot promise that $\mathcal{K}$ is closed for more than finite (non-empty) unions, i.e., if $K_1, K_2 \in \mathcal{K}$ then $K_1 \cup K_2 \in \mathcal{K}$. Write A° and A^{-} for the interior and closure of $A \subseteq S$.

Definition 2.1: Let $A \subseteq S$. The **saturation** of A is the set

$$\operatorname{sat} A = \bigcap \{G \in \mathcal{G} : A \subseteq G\},$$

and A is called **saturated,** if $A = \operatorname{sat} A$. ∎

By part (a) of the following lemma, $\operatorname{sat}\operatorname{sat} A = \operatorname{sat} A$, so $\operatorname{sat} A$ is saturated. The lemma moreover shows that every subset of a T_1-space is saturated. Its proof is simple, hence omitted.

LEMMA 2.2: *Let $A \subseteq S$, $G \in \mathcal{G}$ and $s \in S$. Then the following three equivalences hold:*

(a) $A \subseteq G$ if and only if $\operatorname{sat} A \subseteq G$;
(b) $s \in \operatorname{sat} A$ if and only if $\{s\}^{-} \cap \operatorname{sat} A \neq \emptyset$; and
(c) A is saturated if and only if $s \in A$ follows from $\{s\}^{-} \cap \operatorname{sat} A \neq \emptyset$.

Also the proof of the next result can safely be omitted.

LEMMA 2.3: *Let $K \subseteq S$. Then $K \in \mathcal{K}$ if and only if $\operatorname{sat} K \in \mathcal{K}$.*

Write $\operatorname{sat}\mathcal{K} = \{\operatorname{sat} K : K \in \mathcal{K}\}$. Lemma 2.3 tells us that $\operatorname{sat}\mathcal{K} \subseteq \mathcal{K}$, so $\operatorname{sat}\mathcal{K}$ equals the collection of compact saturated subsets of S. By a previous remark, $\operatorname{sat}\mathcal{K} = \mathcal{K}$ if S is T_1.

We now single out a very important class of sets in S.

Definition 2.4: A set $I \subseteq S$ is said to be **irreducible,** if whenever $I \subseteq F \cup H$ for some $F, H \in \mathcal{F}$, we have $I \subseteq F$ or $I \subseteq H$. ∎

Write $\mathcal{I}$ for the collection of all non-empty irreducible closed sets in S. It is obvious that all singleton closures are irreducible (i.e., $\{s\}^{-} \in \mathcal{I}$ for all $s \in S$). The definition of a sober space goes in the converse direction.

Definition 2.5: A topological space S is called **sober** if every non-empty irreducible closed subset is the singleton closure of a unique point. ∎

That is, S is defined to be sober if the mapping $s \mapsto \{s\}^-$ is a bijection between S and $\mathcal{I}$. Thus any sober S may be embedded into $\mathcal{F}$, and identified in $\mathcal{F}$ with $\mathcal{I}$. Note moreover that any sober space is T_0 (which is just another way of saying that the mapping above is an injection).

It is a nice exercise to show that an irreducible closed subset of a Hausdorff space cannot contain more than one point. Thus all Hausdorff spaces are sober.

We next define random closed sets in S.

Definition 2.6: An $\mathcal{F}$-valued mapping φ, defined on some probability space Ω, will be said to be a **random closed set** in S, if

$$\{\varphi \cap G \neq \emptyset\} := \{\omega \in \Omega : \varphi(\omega) \cap G \neq \emptyset\}$$

is a measurable event for all $G \in \mathcal{G}$. ∎

It may be a good idea to skip the remaining part of this section and instead refer back to it when the need arises.

We next recall some basic facts for continuous posets. Our main reference is Lawson [8]. The monograph Gierz, Hofmann, Keimel, Lawson, Mislove and Scott [4] contains a lot of important information. Hofmann and Mislove [5] discusses the connection between local compactness and continuity for posets. See also the survey article Lawson [9].

Let now L be a poset. Denote its order by $\leq$. For $x \in L$, write $\uparrow x = \{y \in L : x \leq y\}$ and $\downarrow x = \{y \in L : y \leq x\}$.

Definition 2.7: We say L is a **semi-lattice** if every pair $x, y \in L$ has a greatest lower bound $x \wedge y \in L$. A semi-lattice L is said to be a **lattice** if every pair $x, y \in L$ has a least upper bound $x \vee y \in L$. ∎

Definition 2.8: A non-empty set $D \subseteq L$ is called **directed,** if for all $x, y \in D$, $D \cap \uparrow x \cap \uparrow y \neq \emptyset$. **Filtered** sets are analogously defined by replacing $\uparrow$ with $\downarrow$, and by a **filter** on L, we mean a non-empty set $F \subseteq L$, which is filtered and **upper** in the sense that $\uparrow x \subseteq F$ whenever $x \in F$.

We call L **up-complete** if every directed set $D \subseteq L$ has a least upper bound $\bigvee D \in L$. ∎

Now take L up-complete.

Definition 2.9: Let $x, y \in L$. Say x is **way below** y, and write $x \ll y$, if for every directed $D \subseteq L$, with $y \leq \bigvee D$, we have $x \leq d$ for some $d \in D$. ∎

Definition 2.10: If for each $x \in L$, the set $\{y \in L : y \ll x\}$ is directed and its least upper bound is x, then L is said to be a **continuous poset.** A **continuous (semi-) lattice** is a (semi-) lattice which is continuous. ∎

Gierz et al. [4] have a slightly narrower definition of a continuous lattice. A continuous lattice in our sense is a continuous lattice in their sense if it contains a least member.

Definition 2.11: The continuous poset L is said to be **countably based** if there exists a countable set $Q \subseteq L$, such that $x \ll y$ implies the existence of a $z \in Q$ with $x \leq z \leq y$. ∎

If S is locally compact and second countable, then its collection $\mathcal{G}$ of open sets is a countably generated continuous lattice. This is easy to see.

Let now L be a continuous poset.

Definition 2.12: A set $U \subseteq L$ is said to be **Scott open,** if it is upper and $D \cap U \neq \emptyset$ whenever $D \subseteq L$ is directed and $\bigvee D \in U$. An **open filter** is a filter which is Scott open. ∎

The collection of Scott open sets in L is denoted Scott L. Its subcollection of open filters is denoted OFilt L. It is easy to see that OFilt L is up-complete. Lawson [8, Section 3] characterizes its way below relation, and shows that OFilt L is a continuous poset. Moreover, a set $\mathcal{U} \subseteq \text{OFilt}\, L$ is an open filter if and only if there is a unique $x \in L$ such that $\mathcal{U} = \{F \in \text{OFilt}\, L : x \in F\}$. So OFilt OFilt L is isomorphic to L. This is part of the so called Lawson duality.

It is not hard to see that Scott L is closed for finite intersections and arbitrary unions, thus is a topology on L.

Definition 2.13: The **Scott topology** on L is the topology formed by the Scott open sets. ∎

We also need some information on random variables in L.

Definition 2.14: By a **random variable** in L, we mean an L-valued mapping ξ defined on some probability space Ω, such that $\{x \leq \xi\}$ is measurable for all $x \in L$. ∎

It is easy to see that $\xi : \Omega \to L$ is a random variable if and only if $\{\xi \in F\}$ is measurable for all $F \in \operatorname{OFilt} L$, provided that L is countably generated (cf. Norberg [13]).

If $\mathcal{G}$ is a continuous lattice, then so is $\mathcal{F}$ w.r.t. the exclusion order $\supseteq$. A random closed set in S is a random variable in $\mathcal{F}$ and conversely. This is obvious from the definitions. Many of the results for random closed sets in locally compact, second countable and sober spaces, have generalizations to random variables in countably generated continuous posets. (See Norberg [13].)

We end this section with a description of the result of Strassen [19], on which our first proof of Theorem A is based. Let X and Y be two measurable spaces. Let $\simeq$ be a relation on $X \times Y$. Recall that its **graph** is the set $\{(x, y) \in X \times Y : x \simeq y\}$. First, however, we generalize the notation $\subseteq_{\mathrm{st}}$.

Definition 2.15: Consider two random elements ξ and η in X and Y. They need not be defined on the same probability space. By a **coupling** of ξ and η, we mean a random element $(\hat{\xi}, \hat{\eta})$ in $X \times Y$ (w.r.t. the product σ-field) satisfying $\hat{\xi} \stackrel{\mathrm{d}}{=} \xi$ and $\hat{\eta} \stackrel{\mathrm{d}}{=} \eta$. We let $\xi \simeq_{\mathrm{st}} \eta$ denote the existence of a coupling $(\hat{\xi}, \hat{\eta})$ of ξ and η satisfying $\hat{\xi} \simeq \hat{\eta}$ a.s. ∎

Recall that a topological space is called **Polish** if there is a complete and separable metric generating its topology.

THEOREM B (Strassen): *Let X be a compact Polish space, partially ordered by the relation $\leq$, and let ξ and η be two random elements in X. Suppose the graph of $\leq$ is closed in the product topology. Then $\xi \leq_{\mathrm{st}} \eta$ if and only if*

$$Eh(\xi) \leq Eh(\eta)$$

for non-negative bounded functions h, which are continuous and increasing in the usual sense that $h(x) \leq h(y)$ for $x, y \in X$, $x \leq y$.

Proof: See Ligget [10, p. 72]. ∎

3. Proof of Theorem A and a counter example

Recall that S is assumed to be a locally compact second countable topological space, and that ψ and φ are two given random closed sets in S. Assume that

$$P\bigcap_{i=1}^{n}\{\psi\cap G_i\neq\emptyset\}\leq P\bigcap_{i=1}^{n}\{\varphi\cap G_i\neq\emptyset\} \tag{2}$$

holds true for $n=1,2,\ldots$ and $G_i\in\mathcal{G}$. (This is clearly a consequence of $\psi\subseteq_{\mathrm{st}}\varphi$.)

Hofmann and Mislove [5, Section 1] show that $\mathcal{G}$ is order-isomorphic to the topology of a sober space, so we may (and will) assume below that S is sober. Let us recall from Section 2. that all Hausdorff spaces are sober, and that $\operatorname{sat}\mathcal{K}=\mathcal{K}$ if S is T_1, in particular Hausdorff.

Our first task is to show that

$$P\bigcap_{i=1}^{n}\{\psi\cap K_i\neq\emptyset\}\leq P\bigcap_{i=1}^{n}\{\varphi\cap K_i\neq\emptyset\} \tag{3}$$

holds true for $n=1,2,\ldots$ and $K_i\in\mathcal{K}$. (For Hausdorff-S, this is easy.) We rely on results connecting locally compact spaces with spaces having a continuous topology. Inequality (3) follows directly from the following two lemmata (which are standard facts if S is Hausdorff).

LEMMA 3.1: *Suppose that S is locally compact, second countable and sober. If $K\in\operatorname{sat}\mathcal{K}$, then there are $K_1,K_2,\ldots\in\operatorname{sat}\mathcal{K}$ such that $K_{n+1}\subseteq K_n^{\circ}$ for all n, and $K_n\downarrow K$.*

LEMMA 3.2: *Suppose that S is locally compact, second countable and sober. Take $K,K_1,K_2,\ldots\in\operatorname{sat}\mathcal{K}$ and $G\in\mathcal{G}$. If $K_n\downarrow K\subseteq G$, then $K_n\subseteq G$ for some sufficiently large n.*

Proofs: The collection $\mathcal{G}$ is a countably based continuous lattice. Refer to Hofmann and Mislove [5, Section 2] for a proof of the fact that $\operatorname{sat}\mathcal{K}$ is a continuous semi-lattice w.r.t. the exclusion order, which is isomorphic to the collection of open filters on $\mathcal{G}$. Conclude by Norberg [13, Proposition 3.1] that $\operatorname{sat}\mathcal{K}$ is countably based. Lemma 3.1 now follows by combining the fact that $K_1\ll K_2$ in $\operatorname{sat}\mathcal{K}$, if, and only if, $K_2\subseteq K_1^{\circ}$, with Lawson's [8, Proposition 2.2] characterization of the way below relation. Lemma 3.2 follows since any set of the form $\{K\in\operatorname{sat}\mathcal{K}:K\subseteq G\}$ is an open filter if $G\in\mathcal{G}$. ∎

The collection of all sets of the form

$$\{F \in \mathcal{F} : F \cap K = \emptyset\},$$

where $K \in \mathcal{K}$, is closed for finite intersections (since $\mathcal{K}$ is closed for finite unions). Thus it may serve as a base for a topology on $\mathcal{F}$, which we shall refer to as the **decreasing** topology. One reason for this choice of terminology is the necessity of the first implication of the following lemma.

LEMMA 3.3: *Suppose S is locally compact, second countable and sober. Let $\mathcal{U} \subseteq \mathcal{F}$. Then $\mathcal{U}$ is open in the decreasing topology, if, and only if, the following two implications hold true:*

$$\mathcal{F} \ni H \subseteq F \in \mathcal{U} \quad \Rightarrow \quad H \in \mathcal{U};$$
$$\mathcal{F} \ni F_n \downarrow F \in \mathcal{U} \quad \Rightarrow \quad \exists n : F_n \in \mathcal{U}.$$

Proof: The necessity of the two implications is trivial, so let us assume them true. We need only consider the case $S \notin \mathcal{U}$. Suppose $\mathcal{U}$ is not open. Then there is an $F \in \mathcal{U}$ such that if $\{H \in \mathcal{F} : H \cap K = \emptyset\} \subseteq \mathcal{U}$ for some $K \in \mathcal{K}$, then $F \cap K \neq \emptyset$. For each $s \in F^c$ select a pair $(G_s, K_s) \in \mathcal{G} \times \mathcal{K}$ such that $s \in G_s \subseteq K_s \subseteq F^c$. Then $F^c = \bigcup_{s \in F^c} G_s$. By Lindelöf's Theorem (see, e.g., [18]), there are $s_1, s_2, \ldots \in F^c$ such that $F^c = \bigcup_n G_{s_n}$. Hence $\bigcap_{i=1}^n G_{s_i}^c \downarrow F$, so we must have $\bigcap_{i=1}^n G_{s_i}^c \in \mathcal{U}$ for some n. Put $K = \bigcup_{i=1}^n K_{s_i}$. Then $K \in \mathcal{K}$ and $F \cap K = \emptyset$. Take $H \in \mathcal{F}$, $H \cap K = \emptyset$. Then $\bigcup_{i=1}^n G_{s_i} \subseteq K \subseteq H^c$. Thus, $H \subseteq \bigcap_{i=1}^n G_{s_i}^c \in \mathcal{U}$, which implies $H \in \mathcal{U}$. This implies a contradiction from which the sufficiency of the two implications follows. ∎

We give two proofs of the next lemma. The first, which is valid for Hausdorff-S only, uses standard (routine) compactness argumentation. The second, which relies on the already mentioned connection between local compactness and continuity for posets, incidentally also proves Lemma 3.3.

LEMMA 3.4: *Suppose S is locally compact, second countable and sober. The decreasing topology on $\mathcal{F}$ is second countable.*

Proofs: As noted we first consider the particular case of a Hausdorff-S. Let $\mathcal{G}_b \subseteq \mathcal{G}$ be a countable base consisting of open sets with compact closure. Suppose $F \cap K = \emptyset$, where $F \in \mathcal{F}$ and $K \in \mathcal{K}$. Then $K \subseteq F^c$ and we may choose finitely many $G_i \in \mathcal{G}_b$ such that $K \subseteq \bigcup_i G_i \subseteq \bigcup_i G_i^- \subseteq F^c$. This means that $F \cap \bigcup_i G_i^- = \emptyset$, and establishes the result in the Hausdorff case.

In the general case when S need not be Hausdorff, we argue as follows. The collection $\mathcal{F}$ is a countably based continuous lattice. Conclude by Hofmann and Mislove [5, Theorem 2.16] that the decreasing topology is generated by the collection of open filters on $\mathcal{F}$. It is well-known (see, e.g., Lawson [8, Proposition 2.3]) that the open filters generate the Scott topology of $\mathcal{F}$. So the decreasing topology is identical to the Scott topology. Finally, conclude by Norberg [13, Proposition 3.1] that the Scott topology is second countable. ∎

We next provide the collection $\mathcal{F}$ of closed sets in S with the coarsest topology containing the decreasing topology and all sets of the form

$$\{F \in \mathcal{F} : F \cap G \neq \emptyset\},$$

where $G \in \mathcal{G}$, first studied by Fell [3]. In the context of continuous lattices this topology is called the Lawson topology. There the next result is well-known. See Gierz et al. [4, Theorem III.1.10]. Matheron [11, Theorem 1-2-1] treats the Hausdorff case.

PROPOSITION 3.5: *Suppose S is locally compact, second countable and sober. Then $\mathcal{F}$ is a compact Polish space.*

Proof: Fell (op. cit.) shows that $\mathcal{F}$ is a compact Hausdorff space. To see that $\mathcal{F}$ is second countable, use Lemma 3.4 and the fact that S is so. Standard results on metrizability (see e.g. Simmons [18]) now shows that $\mathcal{F}$ is Polish. ∎

Let $\mathcal{U} \subseteq \mathcal{F}$ be open and decreasing in the sense of the first implication of Lemma 3.3. Suppose $\mathcal{F} \ni F_n \downarrow F \in \mathcal{U}$. Then $F_n \to F$. Hence $F_n \in \mathcal{U}$ for sufficiently large n, and we may conclude by Lemma 3.3 that $\mathcal{U}$ is open in the decreasing topology.

By Lemma 3.4,

$$\mathcal{U} = \bigcup_n \{F \in \mathcal{F} : F \cap K_n = \emptyset\}$$

for some $K_1, K_2, \ldots \in \mathcal{K}$. So

$$P\{\psi \notin \mathcal{U}\} = \lim_n P \bigcap_{i=1}^n \{\psi \cap K_i \neq \emptyset\} \leq \lim_n P \bigcap_{i=1}^n \{\varphi \cap K_i \neq \emptyset\} = P\{\varphi \notin \mathcal{U}\},$$

follows by (3).

We have shown that

$$P\{\psi \in \mathcal{H}\} \leq P\{\varphi \in \mathcal{H}\}$$

holds true whenever $\mathcal{H} \subseteq \mathcal{F}$ is closed and its indicator $1_{\mathcal{H}}$ is increasing. By a routine extension, cf. e.g. Kamae, Krengel and O'Brien [6], this implies that $Eh(\psi) \leq Eh(\varphi)$ holds true for all bounded increasing upper semicontinuous non-negative functions h on $\mathcal{F}$.

The next result may be seen as a consequence of the well-known fact that any continuous lattice has closed order. See Gierz et al. [4, Theorem III.2.4 (and III.2.9)]. We give a direct proof.

LEMMA 3.6: *Suppose S is locally compact, second countable and sober. Then the graph of $\subseteq$ is closed in the product topology.*

Proof: Take $(H,F),(H_1,F_1),(H_2,F_2),\ldots \in \mathcal{F}\times\mathcal{F}$, such that $H_n \to H$, $F_n \to F$ and $H_n \subseteq F_n$ for all n. To see that $H \subseteq F$, let $s \notin F$. By local compactness, $s \in G \subseteq K \subseteq F^c$ for some $G \in \mathcal{G}$ and $K \in \mathcal{K}$. Clearly $F \cap K = \emptyset$. Hence $F_n \cap K = \emptyset$ for sufficiently large n. Hence $H_n \cap G = \emptyset$ for infinitely many n, showing $H \cap G = \emptyset$. Hence $s \notin H$. ∎

Now $\psi \subseteq_{st} \varphi$ follows by Strassen's Theorem B. This proves parts (a) and (b) of Theorem A — part (c), which presumes that S is Hausdorff, follows also, since in this case $\mathcal{K} \subseteq \mathcal{F}$. Maybe we should point out that in this case any event of the form $\{\varphi \cap F \neq \emptyset\}$ is measurable if $F \subseteq S$ is closed. See Matheron [11, p. 30].

We conclude the section with a promised counter example.

Example 1: Provide $\{a,b\}$ with the discrete topology (in which all subsets are open) and let the distributions of ψ and φ be given by the probabilities

$$P\{\psi = \emptyset\} = 2/3,\ P\{\psi = \{a,b\}\} = 1/3$$

and

$$P\{\varphi = \{a\}\} = P\{\varphi = \{b\}\} = 3/8,\ P\{\varphi = \{a,b\}\} = 1/4.$$

Then, as the reader easily verifies by direct computation,

$$P\{\psi \cap A \neq \emptyset\} \leq P\{\varphi \cap A \neq \emptyset\}$$

for all $A \subseteq \{a, b\}$, while

$$P\{\psi = \{a,b\}\} > P\{\varphi = \{a,b\}\}.$$

The latter contradicts of course $\psi \subseteq_{\rm st} \varphi$. ∎

4. Some applications of Theorem A

Below S is assumed to be locally compact and second countable. The first result of this section is an immediate consequence of Theorem A.

PROPOSITION 4.1: *Suppose S is locally compact and second countable. Let φ be a random closed set in S, and take $F \in \mathcal{F}$. Then $F \subseteq \varphi$ a.s. if, and only if, $\varphi \cap G \neq \emptyset$ a.s. for all $G \in \mathcal{G}$ with $F \cap G \neq \emptyset$.*

We next look at the thinning result for simple point processes. However in a larger generality. We see S together with its Borel sets (i.e., the σ-field generated by $\mathcal{G}$) as a measurable space. Locally finite measures (i.e., measures μ satisfying $\mu(G) < \infty$ for all relatively compact $G \in \mathcal{G}$) are discussed in Norberg [13]. See also below. For simplicity, we treat now only the Hausdorff case.

Definition 4.2: Let μ be a locally finite measure on S. The set

$$\operatorname{supp} \mu = \{s \in S : s \in G \in \mathcal{G} \Rightarrow \mu(G) > 0\}$$

is called the **support** of μ. ∎

PROPOSITION 4.3: *Suppose S is locally compact, second countable and Hausdorff. Let μ be a locally finite measure on S. Then $\operatorname{supp} \mu \in \mathcal{F}$, and, for all $G \in \mathcal{G}$, $\operatorname{supp} \mu \cap G \neq \emptyset$ if, and only if, $\mu(G) > 0$.*

Proof: Let $s \in (\operatorname{supp} \mu)^c$. Then there is a $G \in \mathcal{G}$ with $s \in G$ satisfying $\mu(G) = 0$. That $G \subseteq (\operatorname{supp} \mu)^c$ is a triviality, so we must have $(\operatorname{supp} \mu)^c \in \mathcal{G}$. We furthermore conclude from the fact that S is second countable, that $\mu((\operatorname{supp} \mu)^c) = 0$. ∎

Let ξ be a random measure in S. Proposition 4.3 shows that $\operatorname{supp} \xi$ is a random closed set. This proposition combined with Theorem A now immediately yields the following result.

THEOREM 4.4: *Suppose S is locally compact, second countable and Hausdorff. Let ξ and η be random measures in S. Then* $\operatorname{supp}\xi \subseteq_{st} \operatorname{supp}\eta$ *if, and only if,*

$$P\bigcap_i\{\xi(G_i)>0\} \le P\bigcap_i\{\eta(G_i)>0\}$$

whenever $\{G_i\}$ is a finite collection of open sets.

The existence result for so called realizable thinnings of simple point processes in locally compact second countable Hausdorff spaces mentioned in the introduction is of course a simple particular case of Theorem 4.4. There is no need for any details.

Following Artstein [1], we make the following definition.

Definition 4.5: Let φ a random closed set in S. We say that a probability measure μ on S is **selectionable** w.r.t. (the distribution) of φ (or φ-**selectionable**) if there exist on some probability space a random element $\hat{\xi}$, distributed according to μ, and a random closed set $\hat{\varphi}$, with the same law as φ, such that $\hat{\xi} \in \hat{\varphi}$ a.s. ∎

That is, a random element ξ in S has a φ-selectionable distribution if, and only if, $\xi \in_{st} \varphi$. The next result is due to Artstein (op. cit.) for Hausdorff-S and extended to all Hausdorff regular spaces by Ross [17].

THEOREM 4.6: *Suppose S is locally compact, second countable and sober. Let μ be a probability measure on S and let φ be a random closed set in S. Then μ is φ-selectionable if, and only if,*

$$\mu(B) \le P\{\varphi\cap B \ne \emptyset\} \tag{4}$$

holds true for all $B \in \mathcal{G}$, if, and only if, it holds true for all $B \in \operatorname{sat}\mathcal{K}$.

Proof: If μ is φ-selectionable, then, clearly, (4) holds for compact saturated sets. But then it holds for open sets, too, as the reader easily sees. So assume (4) true for all $B \in \mathcal{G}$. Let ξ be a random element in S with distribution μ. Put $\psi = \{\xi\}^-$. Then $\psi \cap G \ne \emptyset$ if, and only if, $\xi \in G$, for $G \in \mathcal{G}$, so ψ is a random closed set. Let $G_1, \ldots, G_n \in \mathcal{G}$. By assumption,

$$P\bigcap_i\{\psi\cap G_i\ne\emptyset\} = P\{\xi\in\bigcap_i G_i\} \le P\{\varphi\cap\bigcap_i G_i \ne\emptyset\} \le P\bigcap_i\{\varphi\cap G_i\ne\emptyset\}$$

and $\psi \subseteq_{st} \varphi$ follows by (a) of Theorem A. That is, there is a coupling $(\hat{\psi}, \hat{\varphi})$ of ψ and φ satisfying $\hat{\psi} \subseteq \hat{\varphi}$ a.s. Now $\hat{\psi} \stackrel{d}{=} \psi$, so $\hat{\psi}$ is a.s. non-empty and irreducible. By sobriety, there is a random element $\hat{\xi}$ in S, satisfying $\{\hat{\xi}\}^- = \hat{\psi}$. Then $\hat{\xi} \in \hat{\psi}$, so $\hat{\xi} \in \hat{\varphi}$ a.s. ∎

Definition 4.7: Suppose S is T_0. Let $s, t \in S$. We write $s \leq t$ is $s \in \{t\}^-$. This relation on $S \times S$ is called the **order of specialization.** ∎

Clearly $s \leq t$ if, and only if, $\{s\}^- \subseteq \{t\}^-$, showing that the order of specialization is a partial order on S if S is T_0. Here is the result from which Strassen's Theorem B follows. Note that it says nothing in the Hausdorff case.

PROPOSITION 4.8: *Suppose S is locally compact, second countable and sober, and let ξ and η be random elements in S. Then $\xi \leq_{st} \eta$ if, and only if, the inequality*

$$P\{\xi \in B\} \leq P\{\eta \in B\}$$

holds true for all $B \in \mathcal{G}$, if, and only if, it holds true for all $B \in \operatorname{sat} \mathcal{K}$.

Proof: If the inequality holds true for all $B \in \mathcal{G}$, then, by Theorem 4.6, $\xi \in_{st} \{\eta\}^-$ and $\xi \leq_{st} \eta$ follows by sobriety (argue as in the proof of Theorem 4.6). The remaining equivalence is straightforward. ∎

More can be said in cases where we have good knowledge of the collections $\mathcal{G}$ and/or $\operatorname{sat} \mathcal{K}$. For instance when a convenient base for the topology is given. We now consider an important example.

By Lawson [8, Proposition 5.2], the Scott topology makes any continuous poset L locally compact, sober. Moreover, the Scott topology is second countable if L is countably based (cf. Norberg [13, Proposition 3.1]). It is easy to see that the original order on L coincides with the specialization order.

THEOREM 4.9: *Let ξ and η be random variables in a countably generated continuous poset L. The following three statements are equivalent:* (a) $\xi \leq_{st} \eta$; (b) *the inequality*

$$P\{\xi \in \bigcup_{i=1}^{n} F_i\} \leq P\{\eta \in \bigcup_{i=1}^{n} F_i\}$$

holds true for $n = 1, 2, \ldots$ and $F_i \in \operatorname{OFilt} L$; and (c) *the inequality*

$$P \bigcup_{i=1}^{n} \{x_i \leq \xi\} \leq P \bigcup_{i=1}^{n} \{x_i \leq \eta\}$$

holds true for $n = 1, 2, \ldots$ *and* $x_i \in L$.

Proof: The equivalence between (a) and (b) follows at once from Proposition 4.8, and the fact that $\operatorname{OFilt} L$ is a base for the Scott topology (see Lawson [8, Proposition 2.3]). That (a) implies (c) is obvious, so we need only consider the implication from (c) to (b). This however is a straightforward consequence of the fact that whenever $F \in \operatorname{OFilt} L$, there are $x_1, x_2, \ldots \in L$ such that $x_{n+1} \ll x_n$ for all n, and $(\uparrow x_n) \uparrow F$. This follows by Lawson [8, Propositions 2.2 and 2.4], since L is countably based. ∎

We next apply Theorem 4.9 to random capacities.

Definition 4.10: By a random capacity on S we shall mean a stochastic process ξ, indexed by $G \in \mathcal{G}$ and taking values in $[0, \infty]$, satisfying with probability one the following three conditions: (i) $\xi(\emptyset) = 0$; (ii) $\xi(G_1) \le \xi(G_2)$ whenever $G_1, G_2 \in \mathcal{G}$, $G_1 \subseteq G_2$; and (iii) $\xi(G_n) \to \xi(G)$ whenever $G_1, G_2, \ldots \in \mathcal{G}$, $G_n \uparrow G$. ∎

This approach to random capacities extends Norberg [12], in which S in addition is assumed to be Hausdorff. We proceed to show that the random capacities on S are random variables in a continuous lattice of capacities.

Definition 4.11: Call $c : \mathcal{G} \to [0, \infty]$ a **capacity** on S, if c is increasing, $c(\emptyset) = 0$ and $c(G_n) \uparrow c(G)$ as $G_n \uparrow G$. ∎

The collection of all capacities will be denoted $\mathcal{C}$. We order $\mathcal{C}$ by letting $c_1 \le c_2$, if $c_1(G) \le c_2(G)$ for all $G \in \mathcal{G}$. Gierz et al. [4, Theorem II.2.8] show that $\mathcal{C}$ is a continuous lattice. That $\mathcal{C}$ is countably based is easy to see. Cf. also Norberg and Vervaat [14, Section 6].

Whenever $c \in \mathcal{C}$, we put

$$c(K) = \inf\{c(G) : G \in \mathcal{G},\ G \supseteq K\}$$

for $K \in \mathcal{K}$. It is clear that $c(K) = c(\operatorname{sat} K)$, so we may just as well restrict the above definition to $\operatorname{sat} \mathcal{K}$. It is easy to see that

$$c(G) = \sup\{c(K) : K \in \operatorname{sat} \mathcal{K},\ K \subseteq G\}$$

for $G \in \mathcal{G}$.

By definition a $\mathcal{C}$-valued mapping ξ is measurable if, and only if, the event $\{c \leq \xi\}$ is measurable for all $c \in \mathcal{C}$. We now show that this holds if, and only if, $\xi(G)$ is measurable for all $G \in \mathcal{G}$.

PROPOSITION 4.12: *Let ξ be a $\mathcal{C}$-valued mapping defined on a probability space. Then ξ is a random capacity if, and only if, ξ is a random variable in $\mathcal{C}$.*

Proof: We have already noted that we may assume without loss of generality that S is sober. Fix $x \geq 0$. Take $K \in \operatorname{sat}\mathcal{K}$. Let $c_K(G) = x$ if $K \subseteq G$, $= 0$ otherwise. Then $c_K \in \mathcal{C}$ and, moreover, $c_K \leq \xi$ if, and only if, $x \leq \xi(K)$.

This shows that if ξ is a random variable in $\mathcal{C}$, then $\xi(K)$ is a random variable for all $K \in \operatorname{sat}\mathcal{K}$. It is easy to see that this implies that $\xi(G)$ is a random variable for all $G \in \mathcal{G}$.

To see the converse, assume $\xi(G)$ is a random variable for all $G \in \mathcal{G}$. Fix $c \in \mathcal{C}$. If $c(G) \leq \xi(G)$ for all $G \in \mathcal{G}_b$, where $\mathcal{G}_b$ is a base for the topology which is closed for finite unions, then $c \leq \xi$. (This is obvious.) Hence the event $\{c \leq \xi\}$ is measurable. ∎

We next look at some special types of capacities.

Definition 4.13: We call $c \in \mathcal{C}$ **locally finite,** if $c(G) < \infty$ whenever $G \in \mathcal{G}$, $G \subseteq K$ for some $K \in \mathcal{K}$, i.e., is relatively compact. A locally compact $c \in \mathcal{C}$ is called **modular,** if

$$c(G_1 \cup G_2) + c(G_1 \cap G_2) = c(G_1) + c(G_2)$$

for all relatively compact $G_1, G_2 \in \mathcal{G}$. ∎

Norberg [13] shows that a locally finite capacity on S extends to a unique locally finite measure if, and only if, it is modular.

Definition 4.14: By a **sup measure** we mean a $c \in \mathcal{C}$ satisfying

$$c(G_1 \cup G_2) = c(G_1) \vee c(G_2)$$

for all $G_1, G_2 \in \mathcal{G}$. ∎

Vervaat [20] shows that any sup measure c can be represented by a unique upper semicontinuous function g in the following way: $c(G) = \sup_{s \in G} g(s)$ for $G \in \mathcal{G}$. The function g is given by $g(s) = \inf_{s \in G \in \mathcal{G}} c(G)$ for $s \in S$.

Definition 4.15: A random capacity is called **locally finite** or **modular,** if it is so a.s. A locally finite modular random capacity will be referred to as a **random measure.** An **extremal process** is a random capacity which a.s. is a sup measure. ∎

Note that the notion of the support of a measure (see Definition 4.2) easily extends to capacities in locally compact, second countable and sober spaces. Proposition 4.3 is still true (with a $c \in \mathcal{C}$ instead of μ), and Theorem 4.4 extends to random capacities ξ and η in such a space. (The same proofs works.)

The next theorem is our extension of Rolski and Szekli's [16, Theorem 1] coupling result for random measures, which we mentioned in Section 1.. In order to understand it we point out that $[0,\infty]^n$ is continuous relative to the coordinatewise order and emphasize that if ξ is a random capacity, then the evaluation $(\xi(G_1),\ldots,\xi(G_n))$ is a random variable in $[0,\infty]^n$ for every $n = 1,2,\ldots$ and $G_1,\ldots,G_n \in \mathcal{G}$. (It is a nice exercise to apply Theorem 4.9 to the case $L = [0,\infty]^n$ for a fixt $n = 1,2,\ldots$)

THEOREM 4.16: *Let S be locally compact, second countable and sober. Consider two random capacities ξ and η on S. Then $\xi \le_{\rm st} \eta$ if, and only if,*

$$(\xi(G_1),\ldots,\xi(G_n)) \le_{\rm st} (\eta(G_1),\ldots,\eta(G_n))$$

for every $n = 1,2,\ldots$ and $G_i \in \mathcal{G}$. If S happens to be sober, this holds if, and only if,

$$(\xi(K_1),\ldots,\xi(K_n)) \le_{\rm st} (\eta(K_1),\ldots,\eta(K_n))$$

for every $n = 1,2,\ldots$ and $K_i \in \operatorname{sat}\mathcal{K}$.

COROLLARY 4.17: *We have $\xi \overset{\rm d}{=} \eta$ if, and only if,*

$$(\xi(G_1),\ldots,\xi(G_n)) \overset{\rm d}{=} (\eta(G_1),\ldots,\eta(G_n))$$

for every $n = 1,2,\ldots$ and $G_i \in \mathcal{G}$.

Proofs: The corollary follows of course from the fact that $\xi \overset{\rm d}{=} \eta$ if, and only if, $\xi \le_{\rm st} \eta$ and $\eta \le_{\rm st} \xi$, and a similar fact for random vectors.

The only if part of the theorem follows from the fact that if $(\hat{\xi},\hat{\eta})$ is a coupling of ξ,η, then so related is the $2n$-dimensional vector $((\hat{\xi}(G_i)),(\hat{\eta}(G_i))$ to

$((\xi(G_i)),(\eta(G_i))$. To see its if part, prove that

$$P\bigcup_{i=1}^{n}\{c_i\le\xi\}\le P\bigcup_{i=1}^{n}\{c_i\le\eta\}$$

for arbitrary $c_i\in\mathcal{C}$ (cf. Theorem 4.9). Argue as in the proof of the corresponding part of Proposition 4.12. ∎

Example 2: Consider for S the non-negative extended reals $[0,\infty]$ with the so called lower topology, whose non-trivial open sets are the intervals of the form $[0,s)$ for $s>0$. (This is a locally compact second countable sober space.) Any capacity c on $[0,\infty]$ can be identified with a right continuous increasing function $g:[0,\infty)\to[0,\infty]$, defined by writing $g(t)=c([0,t])$ for $t\ge0$. (Note that $[0,t]$, for $t\ge0$, is compact saturated and that there are no other compact saturated sets than these.) Similarly, random capacities on $[0,\infty]$ can be identified with right continuous increasing (random) processes.

Let ξ and η be two right continuous increasing processes on $[0,\infty)$. We may conclude by Theorem 4.16 that $\xi\le_{\rm st}\eta$ if, and only if,

$$\xi_{t_1},\dots,\xi_{t_n}\le_{\rm st}\eta_{t_1},\dots,\eta_{t_n}$$

holds true for $n=1,2,\dots$ and $t_i\ge0$.

Kwiecinski and Szekli [7] have given a sufficient condition for $\xi\le_{\rm st}\eta$ in the case when ξ and η are counting (i.e., distribution) functions of two simple point processes on $[0,\infty)$. Their condition is stated in terms of the compensators of ξ and η. ∎

For the remaining part of the section, take S locally compact, second countable and sober. By Hofmann and Mislove [5, Section 2], the collection $\operatorname{sat}\mathcal{K}$ is a continuous semi-lattice w.r.t. the exclusion order. Conclude by Norberg [13, Proposition 3.1] that $\operatorname{sat}\mathcal{K}$ is countably based.

Definition 4.18: By a **random compact saturated** set in S, we mean a mapping φ from some probability space inte the collection $\operatorname{sat}\mathcal{K}$, such that $\{\varphi\cap F\ne\emptyset\}$ is measurable for all $F\in\mathcal{F}$. ∎

The next result follows at once by the remark preceding the above definition. So there is no need for a proof.

PROPOSITION 4.19: *Let S be locally compact, second countable and sober. Let φ be a* sat $\mathcal{K}$*-valued mapping, defined on some probability space. Then φ is a random compact saturated set if, and only if, $\{\varphi \subseteq K\}$ is measurable for all $K \in \operatorname{sat}\mathcal{K}$.*

That is, any random compact saturated set is a random variable in $\operatorname{sat}\mathcal{K}$, and vice versa. The next theorem is a corollary to Theorem 4.9. We omit the obvious proof.

THEOREM 4.20: *Let S be locally compact, second countable and sober. Let ψ and φ be random compact saturated sets in S. The following three statements are equivalent:* (a) $\psi \subseteq_{\mathrm{st}} \varphi$; (b) *the inequality*

$$P\bigcap_{i=1}^{n}\{\psi \cap F_i \neq \emptyset\} \leq P\bigcap_{i=1}^{n}\{\varphi \cap F_i \neq \emptyset\}$$

holds true for $n = 1, 2, \ldots$ and $F_i \in \mathcal{F}$; and (c) *the inequality*

$$P\bigcup_{i=1}^{n}\{\varphi \subseteq K_i\} \leq P\bigcup_{i=1}^{n}\{\psi \subseteq K_i\}$$

holds true for $n = 1, 2, \ldots$ and $K_i \in \operatorname{sat}\mathcal{K}$.

The following corollary is immediate. It extends Artstein [1, Theorem 2.1] giving necessary and sufficient conditions for the existence of probability distributions, which are selectionable w.r.t. a given random compact set in a complete, separable metric space.

THEOREM 4.21: *Let S be locally compact, second countable and sober. Let ξ be a random element and φ a random compact saturated set in S. Then the following three statements are equivalent:* (a) $\xi \in_{\mathrm{st}} \varphi$; (b) *the inequality*

$$P\{\xi \in F\} \leq P\{\varphi \cap F \neq \emptyset\}$$

holds true for $F \in \mathcal{F}$; and (c) *the inequality*

$$P\{\varphi \subseteq K\} \leq P\{\xi \in K\}$$

holds true for $K \in \operatorname{sat}\mathcal{K}$.

5. Strassen's Theorem

We first show how Strassen's Theorem B follow from Proposition 4.8. So let X be a compact Polish space endowed with a partial order with a closed graph. Fix two random elements ξ and η in X. It is clear that if $\xi \leq_{st} \eta$, then

$$P\{\xi \in U\} \leq P\{\eta \in U\} \tag{5}$$

holds for open upper $U \subseteq X$. Every lower semicontinuous increasing function on X is a pointwise limit of an increasing sequence of continuous increasing functions. So (5) follows if

$$Eh(\xi) \leq Eh(\eta)$$

holds true for all non-negative continuous, increasing functions h on X.

The open upper sets in X form a topology on X, called the **upper** topology. It is known that X is locally compact, second countable and sober in this topology. Cf. Gierz et al. [4, pp. 312-313], but see also the discussion in Lawson [9]. The specialization order on X coincides with the original order. This is easy to see. Proposition 4.8 now tells us that (5) for open upper $U \subseteq X$, also is sufficient for $\xi \leq_{st} \eta$.

We conclude this section (and the paper) by giving a proof, that does not appeal to Strassen's Theorem B, of a particular case of Theorem 4.9, from which Theorem A is an immediate consequence.

PROPOSITION 5.1: *Let ξ and η be two random variables in a countably generated continuous lattice L. Then $\xi \leq_{st} \eta$ if, and only if,*

$$P\bigcup_{i=1}^{n}\{x_i \leq \xi\} \leq P\bigcup_{i=1}^{n}\{x_i \leq \eta\}$$

holds true for $n = 1, 2, \ldots$ and $x_i \in L$.

Proof: The necessity is obvious. To see the sufficiency, take finite sets $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq L$, such that $Q = \bigcup_n Q_n$ is separating in the sense that, if $x \ll y$, then there is some $z \in Q$ with $x \leq z \leq y$. (Cf. Definition 2.11.) Put

$$\xi_n = \bigvee\{x \in Q_n : x \leq \xi\}.$$

Then $\xi_1 \leq \xi_2 \leq \cdots \leq \xi$. Suppose $x \ll \xi$. Then $x \leq y \leq \xi$ for some $y \in Q = \bigcup_n Q_n$. Thus $x \leq \xi_n$ for some sufficiently large n, and $\xi_n \uparrow \xi$ follows.

Note that ξ_n is a random variable in L. This follows from

$$\{x \le \xi_n\} = \bigcup_m \bigcup_{x \le x_1 \vee \cdots \vee x_m} \{x_1 \le \xi, \ldots, x_m \le \xi\},$$

where the inner union is over $(x_1, \ldots, x_m) \in Q_n^m$.

Next note that $\downarrow\eta$ is a random Scott closed subset of L, since $\downarrow\eta = \{\eta\}^-$ in the Scott topology and $\{\downarrow\eta \cap U \neq \emptyset\} = \{\eta \in U\}$ for $U \in \operatorname{Scott} L$. By arguing as in the first paragraph, we see that there are simple random Scott closed sets $\psi_1, \psi_2, \ldots$ in L such that $\psi_n \downarrow (\downarrow\eta)$.

Fix n. Let D be the set of all pairs (F, x) such that F is in the range of ψ_n, x is in the range of ξ_n and $x \in F$. For subsets X of the range of ξ_n, write

$$D_X = \{F : (F, x) \in D \text{ for some } x \in X\}.$$

Then, writing $X = \{x_i\}$,

$$P\{\xi_n \in X\} = P\bigcup_i \{\xi_n = x_i\} \le P\bigcup_i \{x_i \le \xi_n\}$$

$$\le P\bigcup_i \{x_i \le \xi\} \le P\bigcup_i \{x_i \le \eta\}$$

$$= P\bigcup_i \{x_i \in \downarrow\eta\} \le P\bigcup_i \{x_i \in \psi_n\} = P\{\psi_n \in D_X\}.$$

By the Allocation Lemma of Pollard [15], there is a coupling $(\hat{\xi}_n, \hat{\psi}_n)$ of ξ_n and ψ_n, satisfying $(\hat{\xi}_n, \hat{\psi}_n) \in D$, i.e., $\hat{\xi}_n \in \hat{\psi}_n$.

The Lawson topology on L is compact Polish (Gierz et al. [4, p. 146]). The Scott topology on L is locally compact, second countable and sober (Lawson [8, Proposition 5.2]). Hence Fell's topology on the collection of Scott closed sets in L is compact Polish (cf. Proposition 3.5). Thus the pairs $(\hat{\xi}_n, \hat{\psi}_n)$ are random elements in a compact Polish space, so there is a subsequence $(\hat{\xi}_{n_k}, \hat{\psi}_{n_k})$ which converges in distribution to some random element $(\hat{\xi}, \hat{\psi})$. It is clear that $\hat{\xi} \stackrel{d}{=} \xi$ and $\hat{\psi} \stackrel{d}{=} \downarrow\eta$, so we have here a coupling of ξ and $\downarrow\eta$. By sobriety, there is a random variable $\hat{\eta}$ in L, satisfying $\hat{\psi} = \downarrow\hat{\eta}$. Clearly, $\hat{\eta} \stackrel{d}{=} \eta$.

Let $x \in L$ and take $F \subseteq L$ Scott closed. Then $x \in F$ if, and only if, $\downarrow x \subseteq F$. The graph of $\subseteq$ is closed. Hence so is the graph of $\in$. By the Portmanteau Theorem (see e.g. Billingsley [2]),

$$P\{\hat{\xi} \le \hat{\eta}\} = P\{\hat{\xi} \in \downarrow\hat{\eta}\} = P\{\hat{\xi} \in \hat{\psi}\} \ge \limsup_k P\{\hat{\xi}_{n_k} \in \hat{\psi}_{n_k}\} = 1.$$

Hence $\xi \leq_{st} \eta$. ∎

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